Main contains 1. Another way to define / compute differential : Curves
2. Submersion, Immersion, Embedding and Submanifolds.

DEF: A Smooth curve on a manifold M is $c: (a,b) \longrightarrow M$, assume $0 \in (a,b)$ and $c(0)=p$. Usually we say c is a curve on M starting at p. The velocity vector $c'(t_0)$ of the curve c at time $t_0 \in (a,b)$ is def as:

$$c'(t_0) = c_*\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{c(t_0)}M$$

也记为: $\frac{dc}{dt}(t_0)$, $\frac{d}{dt}\Big|_{t_0} c$.

注: $c'(t_0)$实际上是 $T_{c(t_0)}M$ 这个切向量场中的一个切向量, 这个切向量是如何得出的呢?
$(a,b) \subseteq \mathbb{R} \xrightarrow{c} M$ 作为 (a,b) 与 M 这两个流形之间的 smooth map, Will naturally induce the ~~linear~~ linear map $T_{t_0}\mathbb{R} \xrightarrow{c_*} T_{c(t_0)}M$ between 2 ~~vecto~~ tangent space
以 $\left\{\frac{\partial}{\partial t}\Big|_{t_0}\right\}$ 为基的一维线性空间

Write $\dot{c}(t)$ as the calculus derivative, we can proof following propositions:

① $c: (a,b) \longrightarrow \mathbb{R}$ be a curve, $c'(t) = \dot{c}(t) \frac{d}{dx}\Big|_{c(t)}$

② $c: (a,b) \longrightarrow M$ is a smooth curve, $(U, x^1, x^2, \dots x^n)$ be a coordinate chart about $c(t)$. Write $x^i \circ c := c^i$ for the ith component of c. Then:

$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \frac{\partial}{\partial x^i}\Big|_{c(t)}$. Which is to say:

relative to the basis $\{\partial/\partial x^i|_p\}$ for $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector $\begin{bmatrix} \dot{c}^1(t) \\ \dot{c}^2(t) \\ \vdots \\ \dot{c}^n(t) \end{bmatrix}$

Now: smooth curve c at p $\longrightarrow$ $c'(0)$ in T_pM

? does the converse version still holds?

That is to say, given an arbitrary $X_p \in T_pM$, can we find a curve c s.t $c'(0) = X_p$?
The answer is true!

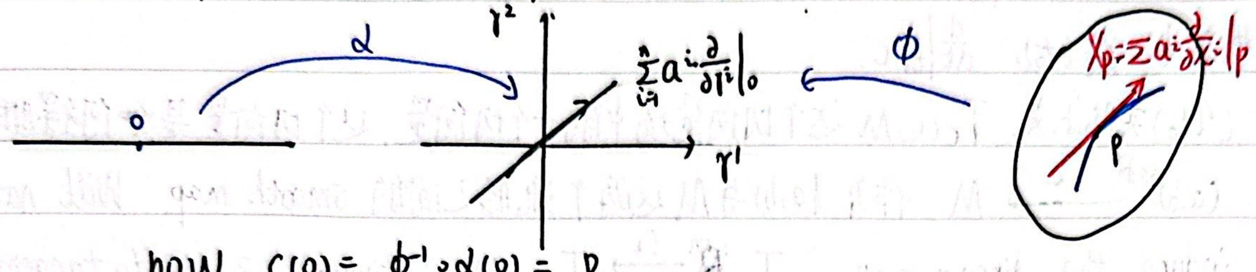
(Existence of a curve with a given initial vector) For any ~~initial~~ $p \in M$, $X_p \in T_pM$, $\exists\, \varepsilon > 0$, with a smooth curve $c: (-\varepsilon, \varepsilon) \longrightarrow M$ s.t. $c(0)=p$, $c'(0)=X_p$.

proof: $(U, \phi) = (U, x^1, x^2, \dots x^n)$ is a chart centered at p. $\phi(p) = 0 \in \mathbb{R}^n$.
设 $X_p = \sum_{i=1}^{n} a^i \partial/\partial x^i|_p$. 令 $\mathbb{R}^n$ 中的标准坐标为 $r^1, r^2, \dots r^n$. 则 $x^i = r^i \circ \phi$
To find a curve c at p with $c'(0) = X_p$:

- Start with a curve α in $\mathbb{R}^n$ with $\alpha(0)=0$, $\alpha'(0)=\sum a^i \frac{\partial}{\partial r^i}\Big|_0$;
- then map α to M via ϕ^{-1}.

We can let $\alpha(t)=(a^1t, a^2t, \cdots a^nt)$, $t\in(-\varepsilon,\varepsilon)$.

$$c=\phi^{-1}\circ\alpha : (-\varepsilon,\varepsilon)\xrightarrow{\alpha}\mathbb{R}^n\xrightarrow{\phi^{-1}}U\subseteq M$$



now $c(0)=\phi^{-1}\circ\alpha(0)=p$.

$$\begin{aligned}
c'(0)&=(\phi^{-1}\circ\alpha)_*\left(\frac{d}{dt}\Big|_{t=0}\right)\\
&=(\phi^{-1})_*\cdot\alpha_*\left(\frac{d}{dt}\Big|_{t=0}\right)\\
&=(\phi^{-1})_*\left(\sum_{i=1}^n a^i\frac{\partial}{\partial r^i}\Big|_0\right)\\
&=\sum_{i=1}^n a^i(\phi^{-1})_*\left(\frac{\partial}{\partial r^i}\Big|_0\right)\\
&=\sum_{i=1}^n a^i\frac{\partial}{\partial x^i}\Big|_p=X_p
\end{aligned}$$

prop: $X_p\in T_pM$, $f\in C_p^\infty(M)$, if $c'(0)=X_p$, then

$$X_pf=\frac{d}{dt}\Big|_0(f\circ c)$$

proof:

$$\begin{aligned}
X_pf&=\left(\sum a^i\frac{\partial}{\partial x^i}\Big|_p\right)(f)\\
&=c_*\left(\frac{d}{dt}\Big|_{t=0}\right)(f)\\
&=\left(\frac{d}{dt}\Big|_{t=0}\right)(f\circ c)\\
&=\text{RHS}.
\end{aligned}$$

Now we use curve to calculate the differentials. (linear maps between tangent spaces)

We have 3 way to compute differential:

① $(F_*(X_p))(f)=X_p(f\circ F)$, $\forall f\in C^\infty_{F(p)}M$

② $(U,x^1,x^2,\cdots x^n)$, $(V,y^1,\cdots y^m)$ 中的natural basis 线性映射在$\left(\frac{\partial}{\partial x^i}\right)$与$\left(\frac{\partial}{\partial y^j}\right)$下的表示矩阵为$\left[\frac{\partial F^i}{\partial x^j}\right]$

③ 用curves (接下来介绍)

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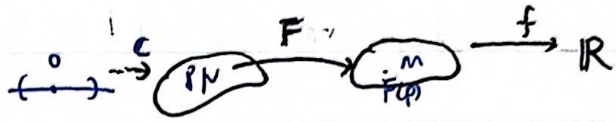
prop: $F: N \to M$ is smooth map of manifolds, $X_p \in T_pN$. if c is a smooth curve starting at p in N with velocity X_p at p, then:

$$F_{*,p}(X_p) = \frac{d}{dt}\Big|_0 (F\circ c)(t).$$

This means: $F_{*,p}(X_p)$ is the velocity vector of the image curve $F\circ c$ at $F(p)$.

proof: $F_{*,p}(X_p) \in T_{F(p)}M$,

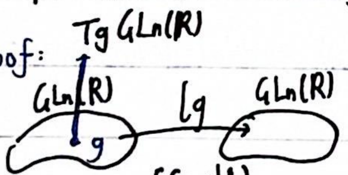


$$\begin{aligned}
F_{*,p}(X_p)(f) &= X_p(f\circ F) \\
&= c'(0)(f\circ F) \\
&= c_*\left(\frac{d}{dt}\Big|_{t=0}\right)(f\circ F) \\
&= \left(\frac{d}{dt}\Big|_{t=0}\right)(f\circ F\circ c) \\
&= \left(\frac{d}{dt}\Big|_{t=0}\right)(F\circ c)(f) \quad \cancel{\Rightarrow RHS.}\ LHS = RHS
\end{aligned}$$

Example (Differential of the left multiplication)

$g \in GL(n,\mathbb{R})$. l_g is the left multiplication on $GL(n,\mathbb{R})$ induced by g. $GL_n(\mathbb{R})$ is an open set in $\mathbb{R}^{n\times n}$, $T_g(GL(n,\mathbb{R}))$ can be identified with $\mathbb{R}^{n\times n}$. Show that this identification of the differential $(l_g)_{*,I}$: $T_I(GL(n,\mathbb{R})) \to T_g(GL(n,\mathbb{R}))$ is also left multiplication by g.

proof:



(Graph 1)

$X \in T_I(GL(n,\mathbb{R})) = \mathbb{R}^{n^2}$. choose $c'(0) = X$.

$$\begin{aligned}
(l_g)_{*,I}(X) &= \frac{d}{dt}\Big|_{t=0} (l_g\circ c(t)) \\
&= \frac{d}{dt}\Big|_{t=0} g\,c(t) \\
&\overset{\text{By } \mathbb{R}\text{-linearity}}{=} g\,c'(0) \\
&= gX
\end{aligned}$$

$$\begin{array}{ccc}
T_I GL_n(\mathbb{R}) & \xrightarrow{(l_g)_{*,I}} & T_g GL_n(\mathbb{R}) \\
\downarrow & & \downarrow \\
I \in GL_n(\mathbb{R}) & \xrightarrow{l_g} & g
\end{array}$$

(Graph 2)

$$\begin{array}{ccccc}
(-\varepsilon,\varepsilon) & \xrightarrow{c} & GL_n(\mathbb{R}) & \xrightarrow{l_g} & GL_n(\mathbb{R}) \\
0 & \longmapsto & I & \longmapsto & g
\end{array}$$

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Immersions and Submersions.

(DEF) C^∞ map $F: N \to M$ is an immersion at $p \in N$ if $F_{*,p}$ is injective.
an submersion $F_{*,p}$ is surjective.

Examples: (prototype of immersion): $\mathbb{R}^n \xhookrightarrow{i} \mathbb{R}^m$. $(x^1, x^2, \cdots x^n) \longmapsto (x^1, x^2, \cdots x^n, 0, 0 \cdots 0)$

(prototype of submersion): $\mathbb{R}^m \xrightarrow{\pi} \mathbb{R}^n$ $(x^1, x^2, \cdots x^m, x^{m+1}, \cdots x^n) \longmapsto (x^1, x^2 \cdots x^m)$

事实上，每个 immersion 局部上就是 inclusion map. 每个 submersion 局部上是 projection.

(Rank): A smooth map $F: N \to M$, its rank at p in N, denote as $rk\, F(p)$, is the rank of the differential $F_{*,p}: T_pN \to T_{F(p)}M$. 也就是 $rk\, F(p) = rk\left[\frac{\partial F^i}{\partial x^j}(p)\right]$

(Critical Point / Regular Point) A point p is critical if $F_{*,p}: T_pN \to T_{F(p)}M$ is not surjective. It is a regular point if $F_{*,p}$ is surjective.

也就是说，p 为 regular point $\Leftrightarrow$ F is a submersion at p.

$h \in M$ is critial value, if it's the image of a critical point. else it's a regular v

注: c is critical value $\Leftrightarrow$ 某些原象中的点 $F^{-1}(f(c))$ 是 critical point.

prop: For a real-valued function $f: M \to \mathbb{R}$, $p \in M$ is critical point $\Leftrightarrow$ relative to some chart $(U, x^1, x^2, \cdots x^n)$ containing p. all partial derivatives satisfy:
$\frac{\partial f}{\partial x^j}(p) = 0$, $j = 1, 2, \cdots n$.

Submanifolds

We will introduce regular submanifolds (locally defined by the vanishing of some of the coordinate functions). The ~~regular set~~ level set theorem tells us that a level set of a C^∞ map of manifolds is a regular submanifold.

We will also introduce constant rank theorem, submersion theorem, actually these can lead to level set theorems simply.

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(regular submanifold of dim k) $S \subseteq$ a manifold N of dim n is regular submanifold of dim k if $\forall p \in S$, $\exists$ a coordinate neighborhood $(U,\phi)=(U,x^1,x^2,\cdots x^n)$ of p in N s.t. $U \cap S$ is defined by the vanishing of $n-k$ of coordinate functions. By renumbering coordinates assume these coordinates $x^{k+1}, x^{k+2}, \cdots x^n$ are vanishing.

- On $U \cap S$, $\phi=(x^1,x^2,\cdots x^k,0,\cdots 0)$. Let $\phi_S: U \cap S \to R^k$ is the restriction of first k components. $(U \cap S, \phi_S)$ is a chart for S in the subspace topology.

(Codimension) S is a regular submanifold of dim k in N of dim n, then $n-k$ is said to be the codimension of S in N.

注1: 以后我们的 submanifold 通常默认指 regular submanifold.

注2: ~~sub~~ regular submanifolds are exactly belong to submanifolds. For $\mathcal{U}=\{(U,\phi)\}$ a collection of compatible ~~adad~~ adapted charts of N that covers S. Then $\{(U \cap S, \phi_S)\}$ is an atlas for S.

(level sets) A level set of map $F: N \to M$ is a subset $F^{-1}(\{c\})=\{p \in N \mid F(p)=c\}$

- c is the level of the level set $F^{-1}(c)$. $Z(F)$ is the zero set of F.

The inverse image $F^{-1}(c)$ of a regular value c is called a regular level set.

(Example): $f(x,y,z)=x^2+y^2+z^2-1=0$. $S^2=f^{-1}(0)$

$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})=(2x,2y,2z)$. Hence the critical point of f is $(0,0,0)$, which doesn't lie on the sphere S^2. Thus, all points on the sphere are regular points of f, 0 is a regular value of f.

Theorem: $g: N \to R$ is a C^∞ function on manifold N. A nonempty regular level set $S=g^{-1}(c)$ is a regular submanifold of N with codimension 1.

We can apply inverse function theorem to prove it.

Given above theorem, we want to extend it to the maps between manifolds. the extended theorem is also know as the implicit function theorem.

KOKUYO

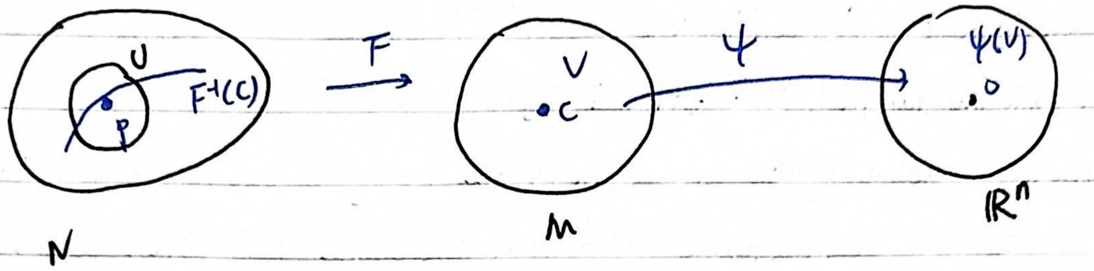
regular level set theorem

Theorem. $F: N \to M$ is a C^∞ map of manifolds. $\dim N = n$, $\dim M = m$. Then a nonempty regular level set $F^{-1}(c)$ (where $c \in M$), is a regular submanifold of N of dimension $n-m$.

pf: $(V, y^1, y^2, \cdots y^m) = (V, \psi)$ 为 c 点附近的坐标卡 (不妨 $\psi(c)=0$). $F^{-1}(V)$ 为 N 中包含了 $F^{-1}(c)$ 之开集.

$F^{-1}(c) = (\psi \circ F)^{-1}(0)$. $F^{-1}(c)$ is the zero set of $\psi \circ F$.

$F^i = y^i \circ F = r^i \circ (\psi \circ F)$, $F^{-1}(c)$ 也就是 $F^1, F^2, \cdots F^m$ 的 common zero point.



Since regular level set $\neq \phi$, we know $n \geq m$.

固定 $p \in F^{-1}(c)$, $(U, x^1, x^2, \cdots x^n)$ 为 coordinate neighborhood of p in N. Since $F^{-1}(c)$ is regular level set, $p \in F^{-1}(c)$ is a regular point. $\Rightarrow$ the $m \times n$ Jacobian matrix $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$ has rk m. WLOG, assume $\left[\frac{\partial F^i}{\partial x^j}\right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$, the first $m \times m$ block, is nonsingular.

We replace the first m coordinates $x^1, x^2 \cdots x^m$ of the chart (U, ϕ) by $F^1, F^2, \cdots, F^m$. $\exists\, U_p \ni p$, s.t $(U_p, F^1, F^2, \cdots F^m, x^{m+1}, \cdots x^n)$ is a chart in the atlas of N.

(原因 $\begin{bmatrix} \frac{\partial F^i}{\partial x^j} & \frac{\partial F^i}{\partial x^\beta} \\ \frac{\partial x^\alpha}{\partial x^j} & \frac{\partial x^\alpha}{\partial x^\beta} \end{bmatrix} = \begin{bmatrix} \frac{\partial F^i}{\partial x^j} & * \\ 0 & I \end{bmatrix}$, $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1 \leq i,j \leq m} \neq 0$.)

In the chart $(U_p, F^1, F^2, \cdots F^m, x^{m+1}, x^{m+2}, \cdots x^n)$, $S = f^{-1}(c)$ is gained by setting the first m coordinate functions $F^1, F^2, \cdots, F^m$ equal to 0. Thus S is a regular submanifold of N of dim $n-m$.

Examples : $SL_n(\mathbb{R})$

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s.t.

$SL(n,\mathbb{R})$ is a regular submanifold of $GL(n,\mathbb{R})$

$f: GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$ be the determinant map. $f^{-1}(1)=SL(n,\mathbb{R})$, f is a C^∞ map between manifolds. We need to show 1 is a regular value.

$f(A)=\det A=(-1)^{i+1}a_{i1}m_{i1}+(-1)^{i+2}a_{i2}m_{i2}+\cdots+(-1)^{i+n}a_{in}m_{in}$.

$$\frac{\partial f}{\partial a_{ij}}=(-1)^{i+j}m_{ij}.$$

Hence $A\in GL(n,\mathbb{R})$ is critical $\Leftrightarrow \frac{\partial f}{\partial a_{ij}}=0 \Leftrightarrow m_{ij}=0$, thus $\det A=0$

$\Rightarrow$ all matrics in $SL(n,\mathbb{R})$ are regular points of the determinant function

$\Rightarrow SL(n,\mathbb{R})$ is a regular submanifold of $GL(n,\mathbb{R})$ of codim 1.

$\dim SL(n,\mathbb{R})=\dim(GL(n,\mathbb{R}))-1=n^2-1$.

Now, we discuss the local structure of a smooth map through its rank.

$f: N\longrightarrow M$, if f has maximal rank at p, there are 3 possibilities:

① $n=m$, by Inverse Function Theorem f is locally diffeomorphism.

② $n\leq m$ $\quad rk(p)=n$. f is an immersion at p.

③ $n\geq m$ $\quad rk(p)=m$. f is ~~an~~ submersion at p.

If f has constant rank on an open set U, we can establish constant rank theorem to give a sample normal form of the smooth map f.

(Constant rank thm) N. M 为两个 n. m 维流形. $f: N\longrightarrow M$ has constant rank k in a neighborhood of a point p in N. $\Rightarrow \exists$ p 点周围的坐标卡 (U,ϕ), $f(p)$ 点周围的坐标卡 (V,ψ). s.t. $(\psi\circ f\circ\phi^{-1})(r^1,r^2,\dots r^n)=(r^1,r^2,\dots r^k,0,0\cdots 0)$

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[d, "\phi(U)"'] & M \arrow[d, "\psi(V)"] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^m \end{tikzcd}$$

证：首先由定义可找到 $\Psi\circ f\circ\Phi^{-1}$ has constant rank k in an open neighborhood of $\Phi(p)$ in $\mathbb{R}^n$.

通过已有的欧氏空间里的常秩定理找到 G (a diffeomorphism of a neighborhood of $\Phi(p)$) and F (a diffeomorphism of a neighborhood of $\Psi(f(p))$). s.t.

$(F\circ\Psi\circ f\circ\Phi^{-1}\circ G^{-1})(r^1,r^2,\dots r^n)=(r^1,r^2,\dots r^k,0,0\cdots 0)$

Immersion and Submersion Thm.

We directly give the outcome.

(Immersion theorem) $f: N \to M$ has an immersion at $p \in N$. then $\exists (U, \phi)$ centred at p in N. (V, ψ) centred at $f(p)$ in M. s.t. in a neighborhood of $\phi(p)$,

$$(\psi \circ f \circ \phi^{-1})(r^1, r^2, \cdots r^n) = (r^1, r^2, \cdots r^n, 0, 0 \cdots 0)$$

(Submersion theorem) $f: N \to M$ is a submersion at p in N. then $\exists (U, \phi)$ centred at p in N, (V, ψ) centered at $f(p)$ in M. s.t in a neighborhood of $\phi(p)$,

$$(\psi \circ f \circ \phi^{-1})(r^1, r^2, \cdots r^m, r^{m+1}, \cdots r^n) = (r^1, r^2, \cdots r^m)$$